Energy budgets in Charney-Hasegawa-Mima and surface quasigeostrophic turbulence

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We study energy transfer in unbounded Charney-Hasegawa-Mima and surface quasigeostrophic turbulence. The possible inverse-cascading quantities in these systems are, respectively, $I\equiv\int_0^\infty k^{-2}E(k)\,dk$ and $J\equiv\int_0^\infty k^{-1}E(k)\,dk$, where $E(k)$ is the kinetic energy spectrum. The supposed direct-cascading quantities for both surface quasigeostrophic and Navier-Stokes turbulence are shown to be bounded. We derive a constraint on $E(k)$ for the surface quasigeostrophic system.

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Two-dimensional (2D) turbulence governed by the Navier-Stokes (NS) equations, the Charney-Hasegawa-Mima (CHM) equation [1–3], and the so-called α turbulence equations [4] is characterized by the simultaneous existence of two inviscid quadratic invariants. For unbounded 2D NS turbulence, the conserved quantities are the kinetic energy $E\equiv\int_0^\infty E(k)dk$ and fluid enstrophy $Z\equiv\int_0^\infty k^2E(k)dk$, where $E(k)$ is the kinetic energy spectrum. For CHM turbulence, the conserved quantities are the total energy $E+\lambda^2 I$ and total enstrophy $Z+\lambda^2 E$, where $I\equiv\int_0^\infty k^{-2}E(k)dk$ and λ is a positive constant. The class of α turbulence features $\int_0^\infty k^{\alpha-2}E(k)dk$ and $\int_0^\infty k^{2\alpha-2}E(k)dk$ as inviscid invariants. Statistical quasiequilibrium arguments [5] show the possible existence of a dual cascade in 2D NS turbulence: an inverse energy cascade to low wave numbers and a direct enstrophy cascade to high wave numbers. These arguments, when applied to the other two cases, imply the possibility of a dual cascade of the corresponding quadratic quantities [6,7].

Interesting cases arise where the supposed direct-cascading quantity is the kinetic energy. This occurs for α turbulence when $\alpha=1$, a model known as the surface quasigeostrophic (SQG) equation. Another case is the CHM equation in the asymptotic limit $\lambda/s\to\infty$, where s is the forcing wave number. The system obtained in this limit is often referred to as the *asymptotic model* (AM) [8,9] or the *potential energy regime* of the CHM system. In the former system, the inviscid invariants are $J\equiv\int_0^\infty k^{-1}E(k)dk$ and E. For the latter case, the inviscid invariants become I and E. This rules out the possibility of the kinetic energy being transferred toward low wave numbers by nonlinear interactions, in marked contrast to 2D NS turbulence, where the kinetic energy is transferred to ever-larger scales at a steady rate. As a consequence, the possibility that the kinetic energy may grow unbounded, as in the NS system, is in jeopardy. In fact, we will establish that the kinetic energy of the SQG system remains bounded. This implies that the spectrum $E(k)$ must be shallower than k^{-1} for $k<s$, which is significantly shallower than that predicted theoretically and found numerically for other systems. Such a constraint may not exist for the CHM system; however, if the kinetic energy grows unbounded, it does so at a vanishing growth rate. This applies for all $\lambda/s>0$, the potential energy regime being a rather peculiar case, where both of the supposed cascading quantities may grow unbounded (a dynamical behavior forbidden in the other systems).

The CHM equation, which governs the potential vorticity of an equivalent-barotropic fluid is [2,8,10]

$$\frac{\partial}{\partial t}(\Delta-\lambda^2)\phi+J(\phi,\Delta\phi)=\nu\Delta^2\phi+f. \tag{1}$$

Here $\phi(x,y,t)$ is the variable part of the free surface height or the stream function of the fluid and f is the forcing. The spatial operators $J(\cdot,\cdot)$ and Δ are, respectively, the Jacobian and two-dimensional Laplacian. The positive constant λ has the dimensions of a wave number (corresponding to the Rossby deformation radius) and ν is the kinematic viscosity coefficient. Equation (1) is known as the quasigeostrophic potential vorticity equation. It also governs the evolution of quasi-2D fluctuations of the electrostatic potential in the plane perpendicular to a strong magnetic field applied uniformly to a plasma, in which case $\phi(x,y,t)$ is the electrostatic potential and λ^{-1} is the ion Larmor radius.

Similar to the NS equations [5,11], the evolution of the ensemble-averaged CHM energy spectrum $E(k)$ is

$$\left(1+\frac{\lambda^2}{k^2}\right)\frac{d}{dt}E(k)=T(k)-2\nu k^2E(k)+F(k). \tag{2}$$

Here $T(k)$ and $F(k)$ are, respectively, the ensemble-averaged energy transfer and energy input. The energy transfer function $T(k)$ is the same as in the NS case:

$$\int_0^\infty T(k)dk=\int_0^\infty k^2T(k)dk=0. \tag{3}$$

As a consequence of these conservation laws, the inviscid unforced dynamics features two quadratic invariants: the total energy $E+\lambda^2 I$ and the total enstrophy $Z+\lambda^2 E$.

In this work, $F(k)$ is assumed to have a spectrally localized support $K=[k_1,k_2]$, where $k_1>0$, and its energy injection $\epsilon\equiv\int_K F(k)dk$ and enstrophy injection $\eta\equiv\int_K k^2F(k)dk$ satisfy $0<k_1^2\epsilon\leqslant\eta\leqslant k_2^2\epsilon$. A hypothesis of this sort is often employed in theoretical studies and is a common numerical situation for the 2D NS system (see Refs. [12,13] and references therein). In fact, this inequality trivially holds for any bandlimited positive injection. We define a forcing wave number $s\equiv\sqrt{\eta/\epsilon}$ which, by hypothesis, lies in K. There are three dynamical regimes, which exhibit quite distinct behaviors. The first is the NS regime obtained when $\lambda/s\to 0$. The

second is the AM model obtained in the limit $\lambda/s \to \infty$. Finally, we have the intermediate regime where the ratio λ/s is finite.

We first review some fundamental dynamical properties of the NS system [5,14–16], which will be compared with those of other regimes and of the SQG system. In the limit $\lambda/s \to 0$, Eq. (2) reduces to the well-known NS energy balance

$$\frac{d}{dt}E(k) = T(k) - 2\nu k^2 E(k) + F(k). \tag{4}$$

Consider an initial injection E_0 at wave number s, corresponding to an enstrophy $Z_0 = s^2 E_0$. The redistribution of E_0 toward $k \gg s$ by nonlinear interactions involves a large increase in the total enstrophy; hence, a direct energy cascade is prohibited. On the other hand, the redistribution of virtually all E_0 toward $k \ll s$ involves a loss of virtually all enstrophy Z_0; hence, to make up for this loss, the remainder of the kinetic energy must be transferred toward $k \gg s$. The spreading of the injected energy from the forcing region into the extremes in this manner is a basis for the classical dual cascade theory of 2D NS turbulence. The energy that gets transferred to $k \ll s$ enjoys virtually no dissipation, while the enstrophy that gets transferred to $k \gg s$ will be completely dissipated. That gives rise to an important feature in this case: the kinetic energy is allowed to grow unbounded in time, but the enstrophy must remain bounded. A simple mathematical basis for this fact can be seen from the evolution of energy and enstrophy:

$$\frac{d}{dt}E = -2\nu Z + \epsilon, \tag{5}$$

$$\frac{d}{dt}Z = -2\nu P + s^2 \epsilon, \tag{6}$$

as obtained by multiplying Eq. (2) by k^2 and integrating both the original and the resulting equations over all wave numbers, noting from the conservation laws (3) that the nonlinear terms drop out. The quantity $P \equiv \int_0^\infty k^4 E(k) dk$ is known as the palinstrophy. Now, if a quasisteady state is established, in which the injection ϵ becomes steady, Z is required to be bounded from above by $\epsilon/(2\nu)$. Otherwise, if $Z > \epsilon/(2\nu)$, the energy would approach zero in the limit $t \to \infty$. On reexpressing P in Eq. (6), we find at sufficiently long times, when E becomes less than $Z/(2s^2)$, that

$$\begin{aligned}\frac{dZ}{dt} &= -2\nu P + s^2\epsilon = 2\nu\left[s^4 E - 2s^2 Z \right.\\ &\quad \left. - \int_0^\infty (k^2 - s^2)^2 E(k) dk \right] + s^2 \epsilon \\ &\leqslant -3\nu s^2 Z + s^2 \epsilon \\ &< -\nu s^2 Z. \end{aligned} \tag{7}$$

This would imply $Z \to 0$ as $t \to \infty$, a clear contradiction. Hence, Z satisfies $Z \leqslant \epsilon/(2\nu)$. However, similar considerations do not apply to the potential energy regime of the CHM equation, making it difficult in that case to establish whether the corresponding direct-cascading quantity E remains bounded.

In the potential energy regime, Eq. (2) reduces to

$$\frac{\lambda^2}{k^2}\frac{d}{dt}E(k) = T(k) - 2\nu k^2 E(k) + F(k), \tag{8}$$

for $k \ll \lambda$. The two inviscid quadratic invariants in Eq. (8) are I and E; the dissipation effectively becomes hyperviscous (bi-Laplacian) and the turbulence evolves on a much slower time scale (by a factor s^2/λ^2). This is apparent from the evolution equations for I and E:

$$\frac{d}{d\tau}I = -\frac{2\nu}{s^2}Z + \frac{\epsilon}{s^2}, \tag{9}$$

$$\frac{d}{d\tau}E = -\frac{2\nu}{s^2}P + \epsilon, \tag{10}$$

where $\tau = (s^2/\lambda^2)t$ is a rescaled time variable. The arguments concerning the turbulent cascade in the NS case apply to this regime, with E and Z replaced by I and E, respectively. More precisely, for an initial injection E_0 at wave number s (corresponding to $I_0 = E_0/s^2$) to spread out in wave number space, the conservation of I and E by nonlinear interactions requires that $I = I_0$ and $E = E_0$ be invariant. Now the redistribution of E_0 toward $k \ll s$ involves a large increase in I; hence, an inverse cascade of E is prohibited. This is analogous to the exclusion of a direct energy cascade in NS turbulence. On the other hand, a redistribution of virtually all of E_0 toward $k \gg s$ involves a loss of virtually all of I_0; hence, to make up for this loss, the remainder of the kinetic energy is required to be transferred toward $k \ll s$. This is analogous to the dual-cascade scenario in NS turbulence, in which the transfer of virtually all of the kinetic energy to $k \ll s$ corresponds to the transfer of the remaining amount to $k \gg s$, due to the conservation of enstrophy. We note, in passing, that although the inverse cascade carries virtually no kinetic energy toward $k = 0$, one cannot, in general, rule out the unboundedness of E in the limit $\tau \to \infty$. This is due to the fact that the dissipation of I is given in terms of Z not E, so a buildup of E (although at a vanishing rate) toward $k = 0$ does not prevent the steady growth of the potential energy.

We now examine the dynamical behavior of the CHM system for a finite ratio λ/s. The evolution of the total energy and enstrophy is governed by

$$\frac{d}{dt}(E + \lambda^2 I) = -2\nu Z + \epsilon, \tag{11}$$

$$\frac{d}{dt}(Z + \lambda^2 E) = -2\nu P + s^2 \epsilon. \tag{12}$$

Similar to the inverse-cascade scenario in NS turbulence, a growth of the total energy requires $2\nu Z<\epsilon$. Such a growth ought to proceed toward low wave numbers (inverse cascade); otherwise, the fluid enstrophy Z would increase until the energy growth rate $\epsilon-2\nu Z$ decreases to zero (for a steady injection rate ϵ). Now an inverse cascade of the total energy involves two unequal parts. For $s<\lambda$, the inverse cascade is predominantly a cascade of potential energy. For $s>\lambda$, the inverse cascade is mainly a cascade of kinetic energy until it arrives at λ. Upon reaching λ, the growth rate is shared equally between the potential and kinetic components. A transition to favor the growth rate of the potential energy occurs when the cascade proceeds to lower wave numbers. To see quantitatively how the kinetic energy growth rate diminishes, we assume that a quasisteady spectrum has been established down to a wave number $k_0<\lambda$ in an ongoing inverse cascade of the total energy. Suppose the spectrum scales as $E(k)=ak^{-\gamma}$ for $k_0\leqslant k<\lambda$. The growth rate of I and E are, respectively, given by

$$\frac{dI}{dt}=\frac{dI}{dk_0}\frac{dk_0}{dt}=-ak_0^{-2-\gamma}\frac{dk_0}{dt}, \tag{13}$$

$$\frac{dE}{dt}=\frac{dE}{dk_0}\frac{dk_0}{dt}=-ak_0^{-\gamma}\frac{dk_0}{dt}. \tag{14}$$

It follows that $dE/dt=(k_0/\lambda)^2\lambda^2 dI/dt$, which diminishes like k_0^2/λ^2, as $k_0/\lambda\to 0$, leaving the potential energy as the only cascading quantity. Equation (13) can be used to estimate k_0. Let $\lambda^2 dI/dt=c\epsilon$, where $0<c<1$, we find $k_0\sim\{a\lambda^2/[c(1+\gamma)\epsilon t]\}^{1/(1+\gamma)}$. If, following Ref. [7], we suppose that $E(k)$ continues to build up on the Kolmogorov spectrum $ak^{-5/3}$ for $k<\lambda$, we would find $k_0\sim[3a\lambda^2/(8c\epsilon t)]^{3/8}$. For comparison we have $k_0\sim[3a/(2c\epsilon t)]^{3/2}$ for the NS case.

For a finite ratio λ/s, one would intuitively expect a compromise between NS and AM dynamics. The arguments in the preceding paragraph suggest that in this intermediate regime, a persistent inverse cascade of the total energy (after the spectrum around λ becomes steady for $s>\lambda$) involves only the potential energy and carries virtually no kinetic energy (or enstrophy). Physically, λ acts as a shield to the kinetic energy with respect to the inverse cascade process [10]. At the same time, a direct cascade of the total enstrophy, if realizable, involves only the fluid enstrophy and carries virtually no kinetic energy (or potential energy). This dynamical behavior is accessible to numerical analysis [10].

There remains the question as to whether the kinetic energy is a finite dynamical quantity. Although the inverse cascade in the region $k<\lambda$ is predominantly a cascade of potential energy, the "leaking" of kinetic energy to ever-larger scales cannot be ruled out. Nevertheless, if E is to become unbounded as $t\to\infty$, it must do so at a vanishing growth rate.

Two special cases of α turbulence, governed by

$$\frac{\partial}{\partial t}(-\Delta)^{\alpha/2}\phi+J(\phi,(-\Delta)^{\alpha/2}\phi)=D(-\Delta)^{\alpha/2}\phi+f,$$

are the Navier-Stokes equations ($\alpha=2$, $D=\nu\Delta$) and the surface quasigeostrophic equation [$\alpha=1$, $D=-\mu(-\Delta)^{1/2}$]. The latter system originates from the geophysical context describing the motion of a stratified fluid with small Rossby and Ekman numbers [4,17,18]. Note that the dissipation operator is hypoviscous; this is the natural physical dissipation mechanism for the SQG system. A viscous dissipation operator $D\propto\Delta$, which would be equivalent to the molecular viscosity in NS turbulence, has also been considered in the literature for numerical purposes [19–21]. The kinetic energy spectrum of this system is studied in Ref. [22], where it is rigorously shown that $E(k)\leqslant ck^{-2}$ for $k<s$, where c is a constant. The inviscid unforced version is known to resemble the 3D Euler equation in many aspects, particularly the possibility of spontaneous development of singularities [23]. This may be attributable to the fact that both systems have similar energy budgets (as shown below, energy is transferred to small scales).

The energy spectrum $E(k)$ evolves according to

$$\frac{d}{dt}E(k)=S(k)-2\mu kE(k)+F(k), \tag{15}$$

where the transfer function $S(k)$ satisfies

$$\int_0^\infty k^{-1}S(k)dk=\int_0^\infty S(k)dk=0. \tag{16}$$

Like the CHM equation in the potential energy regime, nonlinear transfer in this system conserves the kinetic energy as the possible direct-cascading quantity. The supposed inverse-cascading quantity is J. The dissipation of J is given in terms of E, in analogy to the NS system, where the dissipation of energy is given in terms of enstrophy. Therefore, we can invoke the arguments leading to the boundedness of enstrophy in the NS case to conclude that the kinetic energy is bounded in the SQG system. To this end, we consider the evolution of J and E governed by

$$\frac{d}{dt}J=-2\mu E+\frac{\epsilon}{s}, \tag{17}$$

$$\frac{d}{dt}E=-2\mu\int_0^\infty kE(k)dk+\epsilon, \tag{18}$$

where $s\equiv\int_K kF(k)dk/\int_K k^{-1}F(k)dk\in K$ and a corresponding localization hypothesis for the forcing is assumed. Namely, we require $0<k_1\int_K k^{-1}F(k)dk\leqslant\epsilon\equiv\int_K F(k)dk\leqslant k_2\int_K k^{-1}F(k)dk$. Now, if a quasisteady state is established, in which the injection ϵ becomes steady, E is required to be bounded from above by $E\leqslant\epsilon/(2\mu s)$. Otherwise, J would approach zero in the limit $t\to\infty$, which in turn would entail

$$\frac{dE}{dt} = -2\mu\int_0^\infty kE(k)dk + \epsilon = 2\mu\left[s^2J - 2sE - \int_0^\infty (k-s)^2k^{-1}E(k)dk\right] + \epsilon \leqslant -3\mu sE + \epsilon \leqslant -\mu sE, \tag{19}$$

for sufficiently large times. This would imply $E \to 0$ as $t \to \infty$, a clear contradiction. Hence, E is bounded from above by $E \leqslant \epsilon/(2\mu s)$ in a quasisteady state.

The boundedness of E imposes a stiff constraint on $E(k)$, requiring that the spectrum be shallower than k^{-1} for $k<s$. The dimensional analysis prediction $E(k)\sim k^{-1}$ would imply a logarithmic divergence of $\int_0^s E(k)dk$.

We attempted to mimic SQG turbulence in an unbounded domain using a doubly periodic dealiased pseudospectral 2731×2731 simulation and a Cash-Karp Runge-Kutta-Fehlberg temporal integrator. The turbulence was driven from zero initial conditions at time $t=0$ by a random forcing bandlimited to [1198,1202] with velocity amplitude 0.0058. We used the truncated Fourier-transformed dissipation operator $D(k) = -\mu k H(k-1202)$, where H is the Heaviside unit step function and $\mu = 0.054$. While this truncation may in principle break the constraint imposed by the boundedness of E and lead to a steeper slope, one nevertheless obtains the roughly k^{-1} transient spectrum (time averaged from $t=7$ to $t=8$) shown in Fig. 1. This is considerably shallower than the NS $k^{-5/3}$ spectrum and much tighter than the *a priori* estimate $E(k)\leqslant ck^{-2}$ obtained in Ref. [22]. The spike at wave number 1200 is a manifestation of the forcing.

A direct energy cascade requires that the dissipation of energy, which is proportional to $\int_0^\infty kE(k)dk$, be primarily dominated by the high wave numbers, so that $E(k)$ must be at least as shallow as k^{-2} for $k>s$. Otherwise, energy will be dissipated in the vicinity of the forcing wave number s.

For a bounded system in equilibrium, Eqs. (17) and (18) imply $\int_{k_0}^\infty (s-k)E(k)dk = 0$, where k_0 is the lowest wave number corresponding to the system size. Following Ref. [13], if $E(k)$ has the form $E(k)=ak^{-\gamma}$ for $k\leqslant s$ and $E(k) = bk^{-\beta}$ for $s\leqslant k \leqslant k_\nu$, where $s/k_\nu \leqslant k_0/s$, we obtain the constraint $\gamma+\beta\geqslant 3$. In the limits $s/k_\nu \to 0$ and $k_0/s\to 0$, this constraint becomes $\gamma+\beta=3$.

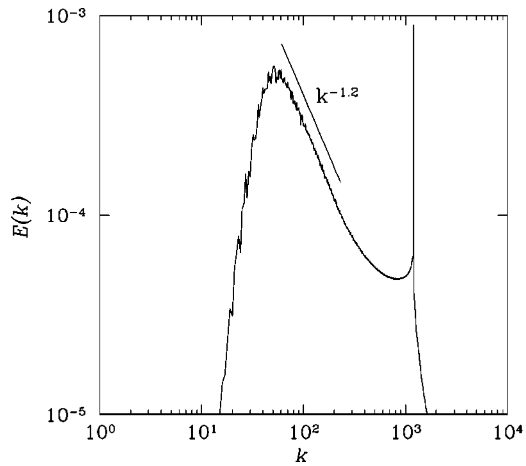

FIG. 1. Inverse cascade of SQG turbulence in the absence of large-scale dissipation.

In conclusion, CHM turbulence (for a general ratio $\lambda/s > 0$) is characterized by an inverse cascade of the potential energy, which carries virtually no kinetic energy, as $t\to\infty$. More precisely, an inverse cascade of kinetic energy is excluded in the region $k<\lambda$. This makes the CHM dynamics distinct from its NS counterpart, although before the inverse cascade reaches λ, CHM turbulence may have much in common with NS turbulence. Similarly, the kinetic energy in the SQG system is prohibited from being transferred to large scales, just like the enstrophy for the NS system. In particular, the kinetic energy is the dissipation agent of the supposed inverse-cascading quantity and is thereby required to be bounded. As a consequence, the energy spectrum is shallower than k^{-1} for $k<s$.

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